1501. A bird of weight W is standing in equilibrium on a roof sloped at 30° to the horizontal. Find the following, in terms of W:

(a) the normal reaction on the bird's feet,

(b) the friction on the bird's feet,

(c) the total contact force on the bird's feet.

1502. A positive polynomial of even degree has single roots at 0 and 1, a double root at 2, and no other roots. Sketch its main features.

1503. The *golden ratio* ϕ is defined as $\phi = \frac{1}{2}(1+\sqrt{5})$. Find, in the form $x^2 + px + q = 0$ for $p, q \in \mathbb{Z}$, the quadratic equation of which ϕ is a root.

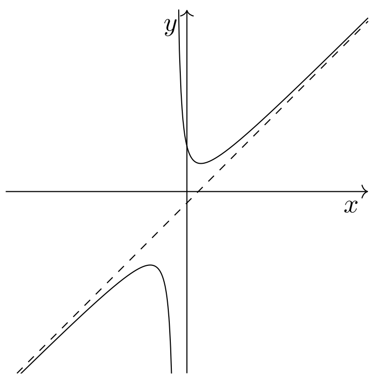
1504. Variables X_1 and X_2 have identical, independent binomial distributions $X_1, X_2 \sim B(4, 0.5)$. State whether the following are distributed binomially; if so, give the distribution in the form $B(n, p)$:

(a) $2X_1$,

(b) $X_1 + X_2$,

(c) $X_1 - X_2$.

1505. The diagram show the curve $y = \dfrac{x^2+4}{x+1}$.



The dashed line is an oblique asymptote.

(a) Find A, B, C such that

$$\frac{x^2+4}{x+1} \equiv Ax + B + \frac{C}{x+1}.$$

(b) Hence, show that the oblique asymptote has equation $y = x - 1$.

1506. A sequence has nth term $u_n = 4n^2 - 18n - 40$, for $n \geq 1$. Find the value of the first non-negative term.

1507. One of the following statements is true; the other is not. Identify and disprove the false statement.

(a) $xyz = 0 \implies xy = 0$,

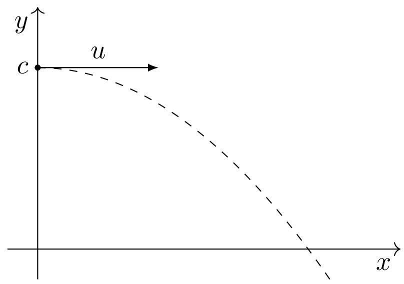
(b) $xy = 0 \implies xyz = 0$.

1508. The result of an experiment is modelled with Y, where Y has a normal distribution with mean 3 and variance 6. Find $\mathbb{P}(Y^2 > 4Y)$.

1509. Separate the variables in the following differential equation, writing it in the form $\text{f}(y)\frac{dy}{dx} = \text{g}(x)$ for some functions f and g:

$$e^{2x+3y}\frac{dy}{dx} = 2.$$

1510. A projectile is launched horizontally from the point $(0, c)$ with speed u ms^{-1}.



Show that the equation of the trajectory is

$$y = c - \frac{gx^2}{2u^2}.$$

1511. Solve for n in $\displaystyle\sum_{k=1}^{n}(2k-1) = 100$.

1512. State, with justification, which of the implications $\implies$, $\impliedby$, $\iff$ could replace the question marks below, linking statements concerning $x \in \mathbb{R}$:

$$0 = x^2 + x + 1 \quad ??? \quad \sqrt{x} = x + 1.$$

1513. Describe the transformation which takes the graph $y = \text{f}(x)$ onto the graph $\text{f}(y) = x$.

1514. Show that the ordinal formula for the sequence $u_n = 4u_{n-1}$, $u_3 = 24$ can be expressed as

$$u_n = 3 \times 2^{2n-3}.$$

1515. Show that it is impossible for a quadratic graph $y = ax^2 + bx + c$, where $a \neq 0$, to pass through the three points $(4, 0)$, $(5, 2)$, and $(7, 6)$.

1516. Give the period of each of the following functions, defined in degrees:

(a) $\sin x$,

(b) $\sin 3x$,

(c) $\sin 2x + \sin 3x$,

(d) $\sin 2x + \sin 4x$.

1517. "An x intercept of $y = \text{f}(x)$ must be an x intercept of $y^2 = \text{f}(x)$." True or false?

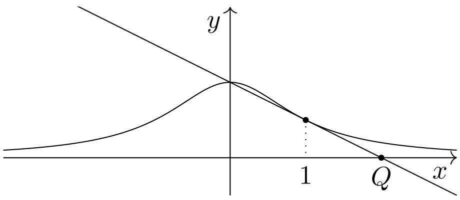
1518. In this question, $X_1 \sim B(n_1, p)$ and $X_2 \sim B(n_2, p)$ are independent binomial variables with the same trial probability. State, with a reason, whether the following variables are binomially distributed. If so, give the distribution in the form $B(n, p)$.

(a) $X_1 + X_2$,
(b) $X_1 X_2$,
(c) $n_1 + n_2 - X_1 - X_2$.

1519. Prove that, if a quadratic graph $y = x^2 + bx + c$ has its vertex at $x > 0$, then $b < 0$.

1520. Show that $\displaystyle\int_0^4 x(x+1)(x+2)\,dx = 144$.

1521. The tangent to the curve $y = \left(1 + x^2\right)^{-1}$ at $x = 1$ crosses the x axis at Q.



Find the coordinates of Q.

1522. A function g has second derivative $\text{g}''(x) = 0$ for all x. The graph $y = \text{g}(x)$ passes through $(-1, 2)$ with gradient 4.

(a) Find $\text{g}(x)$.
(b) Solve the equation $\text{g}(x) = \text{g}(1 - x)$.

1523. A set of data is analysed, and four outliers are then removed from it. State, with a reason, whether the following necessarily hold:

(a) the new IQR is smaller,
(b) the new standard deviation is smaller.

1524. Show that $y = 4x^4 - 8x^2$ has local minima whose y coordinate is -4.

1525. A uniform cylinder of radius r and length l is placed with one of its circular faces on a rough slope of inclination θ. Assuming that friction is high enough so that no slippage occurs, show that the greatest angle of inclination θ on which the cylinder can remain in equilibrium is

$$\theta = \arctan \frac{2r}{l}.$$

1526. You are given that the following curve intersects the x axis at $(k, 0)$:

$$y = \frac{x^2}{1+x} + k.$$

Determine all possible values of k.

1527. Solve $x^{\frac{5}{3}} + 7776 = 275x^{\frac{5}{6}}$.

1528. Two balls of similar size, with masses m and $2m$ kg, are dropped simultaneously from the same height. The air resistance on each ball is modelled as a constant $\frac{1}{4}mg$ N.

(a) Find a formula for the distance d between the two in the ensuing motion.
(b) Find the predicted time taken for this distance to exceed 100 metres.
(c) Give a reason why this value is unlikely to be accurate in any physical scenario.

1529. A set of data, whose mean is 2 and whose standard deviation is 1, is given as follows:

x	0	1	2	3
f	12	14	a	b

Find a and b.

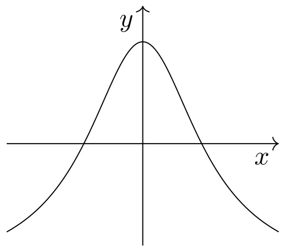
1530. Prove that a quadratic curve and a cubic curve, each of the form $y = \text{f}(x)$, must intersect.

1531. You are given that the equations $x + y = a$ and $2x + by = 10$, for constants $a, b, c \in \mathbb{R}$, have no simultaneous solutions (x, y).

(a) Show that $b = 2$.
(b) Determine all possible values for a, giving your answer in set notation.

1532. The four points A, B, C, D, have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. M is the midpoint of AB, and N is the midpoint of CD. Show that the position vector of the midpoint of MN is the mean of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$.

1533. The diagram shows curve C:



Show that C cannot have equation $y = \dfrac{1}{x^2 + 1}$.

1534. "The y axis is tangent to $(x+1)^2 + (y+1)^2 = 1$." True or false?

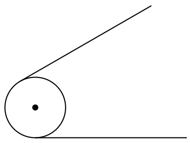
1535. The edges of a regular pentagon are coloured red or blue. Show that, if rotations and reflections are not counted as distinct, there are eight colourings.

1536. Solve $(3x-1)^2 - (3x-1)^3 = 0$.

1537. Determine the number of elements in the set

$$\{x \in \mathbb{Z} : x^2 < 20\} \cap \{x \in \mathbb{Z} : x^3 < 50\}.$$

1538. A light string is passed around a smooth pulley. The sections of string either side of the pulley make an angle 30° with each other, and the tension in the string is $\sqrt{6}$ N.



Show that the force exerted on the pulley by the string is $3+\sqrt{3}$ N. You may assume the following exact value:

$$\cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

1539. State, with a reason, whether the following holds: "A hypothesis test for the probability of success p in a binomial distribution $X \sim B(20,p)$ is to be carried out, testing $H_0 : p = \frac{1}{4}$ and $H_1 : p > \frac{1}{4}$ at the 1% level. Assuming H_0, with c denoting the critical value and x the number of successes,

$$\mathbb{P}(c \leq x) < 0.01 < \mathbb{P}(c-1 \leq x).$$

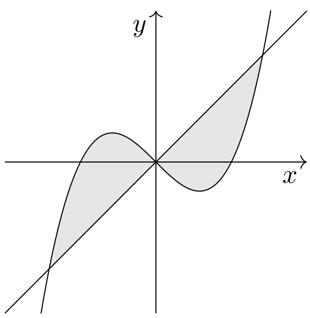
1540. Determine which of the points $(2,-2)$ and $(-6,2)$ is closer to the locus of the equation

$$x^2 + 6x + y^2 + 2y - 20 = 0.$$

1541. By setting up simultaneous equations, find integers p and q such that

$$\left(p\sqrt{3} + q\sqrt{5}\right)^2 = 155 - 40\sqrt{15}.$$

1542. A normal is drawn to the curve $y = x^3 - x$ at the origin. This encloses two regions, which are shown below.



Show that each region has area 1.

1543. "For any quadrilateral, there exists a circle which passes through all four vertices." Disprove this.

1544. Find y in simplified terms of x, if

(a) $3^y = 3^x \cdot 9^x$,

(b) $e^y = e^x \div e^{2x-5}$.

1545. Two invertible functions have $\text{f}(a) = \text{g}(-a) = 0$, and $\text{f}(-a) = \text{g}(a) = -a$. Write down

(a) a root of g,

(b) a fixed point of f,

(c) a root of g^2.

1546. A rectangle has dimensions $x \times y$. Starting from $x = 2$ and $y = 10$, its lengths change at constant rates of 1 and -1 units per second respectively.

(a) Using the product rule, or otherwise, show that the rate of change of area is given by

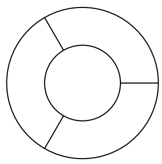
$$\frac{dA}{dt} = y - x.$$

(b) Determine whether, at the beginning, the area increases or decreases.

(c) Find the stationary value of the area.

1547. Solve $\dfrac{x}{x+1} - \dfrac{x-1}{x} = \dfrac{1}{12}$.

1548. The regions of the following diagram are randomly coloured, each red, green or blue.



Write down the probability that no two regions which share a border are coloured the same.

1549. A particle, following a period of time of duration t moving with constant acceleration a, ends up with final velocity v.

(a) Show that the velocity V at time T after the start of the interval of constant acceleration is given by the formula $V = aT + v - at$.

(b) Hence, use definite integration with respect to the variable T to prove that, over this period, the final displacement s is given by

$$s = vt - \tfrac{1}{2}at^2.$$

1550. In each part, sketch a curve of the form $y = \text{f}(x)$ which has the given properties:

(a) positive, increasing and convex everywhere,

(b) positive, decreasing and convex everywhere.

1551. Simplify $\ln\left(2\sqrt{e^n}\right) - \ln 2$.

1552. This question is about the reflection of the cubic $y = x^3 + x^2$ in the line $y = 1$.

(a) By finding any stationary points, show that the cubic crosses the line only once.

(b) Sketch the cubic, the line and the reflection of the cubic on the same set of axes.

(c) By considering reflection in the line $y = 1$ as reflection in $y = 0$ following by translation by some vector $k\mathbf{j}$, determine the equation of the reflected cubic.

1553. To generate the formula for the volume of a sphere of radius r, the following integral is set up

$$V = \int_{-r}^{r} \pi\big(r^2 - x^2\big)\,dx.$$

(a) By considering a circle of radius r in an (x, y) plane, describe the physical significance of the integrand.

(b) Prove that $V = \frac{4}{3}\pi r^3$.

1554. For a quadratic function f, the definite integral of $\mathrm{f}(x)$, between $x = a$ and $x = b$, is maximised when $a = 2$ and $b = 5$. Sketch $y = \mathrm{f}(x)$.

1555. For $x \in [0, 2\pi)$, solve the following equation:

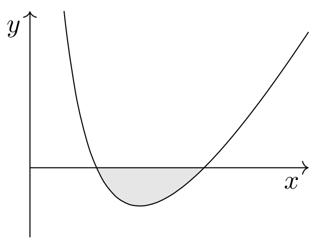
$$\left(\tan x - \tfrac{\sqrt{3}}{3}\right)\left(2\cos x - \sqrt{2}\right) = 0.$$

1556. Explain the (very large!) errors in this appraisal of Newton's Laws:

> For an object in equilibrium, weight and reaction force are equal and opposite. This is due to Newton's third law.

1557. For $\mathrm{g} : x \mapsto \dfrac{2}{\sqrt{3 - 2x}} + 1$, find g^{-1}.

1558. The curve shown below is $y = \dfrac{1}{\sqrt{x}} + x - 2$.



(a) By writing the relevant equation as a cubic in $\sqrt{x}$, show that the curve crosses the x axis at $x = \frac{1}{2}\big(3 - \sqrt{5}\big)$ and $x = 1$.

(b) Hence, find the area of the shaded region.

1559. Either prove or disprove the following statement:

$$\mathbb{P}(A \mid B) > \mathbb{P}(A) \iff \mathbb{P}(A \mid B') < \mathbb{P}(A).$$

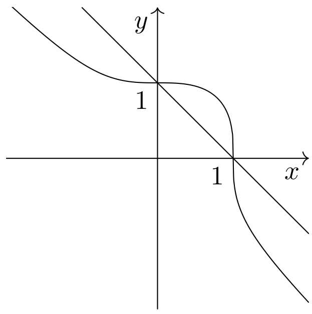
1560. Show that, if x and y are related exponentially by $y = ae^{kx}$, then $\ln y$ and x are related linearly.

1561. You are given that $\mathrm{f}(x) = 2$ has exactly one root, which is $x = p$, and that $\mathrm{f}(x) = 3$ has exactly one root, which is $x = q$. Giving your answers in terms of p and q, solve the following equations:

(a) $\mathrm{f}(4x - 1) = 2$,

(b) $\big(\mathrm{f}(x + 1) - 2\big)\big(2\,\mathrm{f}(x) - 6\big) = 0$.

1562. The graph below shows $x + y = 1$ and $x^3 + y^3 = 1$.



Use the graph to provide counterexamples to the following implications:

(a) $x + y > 1 \implies x^3 + y^3 > 1$,

(b) $x^3 + y^3 > 1 \implies x + y > 1$.

1563. Show that, for any real-valued random variable X, $\mathbb{P}(X^2 + 1 > X) = 1$.

1564. Write $x^4 + x^2 + 1$ in terms of $(x^2 + 1)$.

1565. A particle is moving with constant acceleration in 1D. During the motion, it takes (t, x) values $(0, 2)$, $(2, 4)$ and $(4, 8)$.

(a) Determine the object's acceleration.

(b) Find x in terms of t.

(c) Find the average speed for $t \in [0, 10]$.

1566. The binomial expansion of $(x + a)^5$ is given by $x^5 - 15x^4 + bx^3 + \ldots$, where $a, b \in \mathbb{Z}$. Determine the values of a and b.

1567. A random number generator produces values Y_i distributed normally, with mean and variance 100. A sample of fifty values is taken.

(a) Write down the distribution of $\bar{Y}$.

(b) Find $\mathbb{P}(99 \leq \bar{Y} \leq 101)$.

1568. Sketch $(x + 1)(y + 1) = 1$.

1569. In the solving of a separable differential equation, where the variables x and y are both positive, the following line of algebra has been reached:

$$\int 3x(x+2)\,dx = \int \frac{1}{y}\,dy.$$

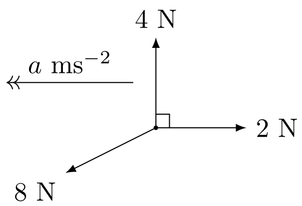
Show that $y = Ae^{x^3+3x^2}$, for some constant $A > 0$.

1570. Describe all functions f for which f′ is quadratic.

1571. Four lines are given as $x - 3y = 0$, $x - 3y = 10$, $3x + y = 10$, and $3x + y = 0$. Show that the lines enclose a square, whose area is 10 square units.

1572. A sequence is defined iteratively as $x_{n+1} = ax_n + b$, for some $a, b \in \mathbb{R}$. Three consecutive terms of the sequence are $8, 25, 93$. Determine a and b.

1573. An object of mass 2 kg has forces acting on it as shown in the diagram. Acceleration is antiparallel to the 2 Newton force.



Determine the obtuse angle between the 8 and 2 N forces, and the exact value of a.

1574. Two of the following statements are true; the other two are not. Disprove the false statements.

(a) $a = b \implies (a-b)(c-d) = 0$,
(b) $a = b \implies (a-b) \div (c-d) = 0$,
(c) $(a-b)(c-d) = 0 \implies a = b$,
(d) $(a-b) \div (c-d) = 0 \implies a = b$.

1575. By considering the distance from the origin, show that no part of the parametric curve $y = \sin t$, $x = \sin 2t$ lies outside the circle $x^2 + y^2 = 2$.

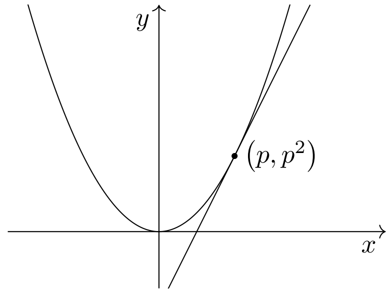
1576. At a surgery, the ages of 10 walk-in patients are recorded. The mean of this data is $\bar{x} = 66.1$ years and the standard deviation is 14.6 years. An overly zealous employee decides to model the ages with a normal distribution.

(a) Find, according to the employee's model, the probability that the next walk-in patient is
  i. under 30 years of age,
  ii. between 50 and 70 years of age.
(b) Give two reasons why these probabilities are unlikely to be useful in making predictions.

1577. Show that, if $x + y > 2$, then $x^2 + y^2 > 2$.

1578. By solving a suitable quadratic equation, factorise the expression $39867x^2 - 93574xy - 57893y^2$.

1579. A tangent T is drawn to the curve $y = x^2$ at the point (p, p^2).



Show that T has equation $y = 2px - p^2$.

1580. Two projectiles are dropped from the same point, with a 1 second delay between them. Prove that, in the subsequent motion, the distance between them grows (theoretically) without bound.

1581. A function f has instruction

$$\mathrm{f} : x \mapsto \ln\left(1 + x + x^2\right).$$

Determine whether the function f is well defined over the domain $\mathbb{R}$.

1582. Three vectors are given as

$$\mathbf{p} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

(a) Write down the angle between $\mathbf{p}$ and $\mathbf{q}$.
(b) The magnitudes of $\mathbf{p}$ and $\mathbf{r}$ are related as $|\mathbf{p}| = k|\mathbf{r}|$. Determine the value of k.

1583. A six-sided die and a twelve-sided die are rolled together. Find the probability that the score on the six-sided die is larger.

1584. Show that, if $y = \tan px$ for some constant p, then

$$\frac{dy}{dx} = p + py^2.$$

1585. A pupil is trying to prove that $\sqrt{2}$ is irrational. He begins with the line "Assume, for a contradiction, that $\sqrt{2}$ cannot be written in the form p/q, where p and q are integers with $\mathrm{hcf}(p, q) = 1$." Explain the error that has been made, and give a corrected opening line.

1586. Using calculus, prove that, for a fixed perimeter P, the rectangle of greatest area is a square.

1587. A function f with the following property is sought:

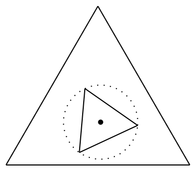
$$\int_0^x \mathrm{f}(t)\,dt \equiv \frac{2}{\mathrm{f}(x)}.$$

(a) A function $\mathrm{f}(x) = x^k$ is proposed. Show that

$$\tfrac{1}{k+1}x^{k+1} \equiv 2x^{-k}.$$

(b) Hence, determine k.

1588. The diagram below shows an equilateral triangle S of unit area, with a smaller equilateral triangle T inside it. T can rotate freely about its centre without intersecting S.



Find the area of the largest such triangle T.

1589. A function f is defined over the reals, and has range $[1, \infty)$. Give the ranges of the following:

(a) $x \mapsto \mathrm{f}(x) + 1$,

(b) $x \mapsto 2\,\mathrm{f}(x) + 1$,

(c) $x \mapsto a\,\mathrm{f}(x) + b$, where $a > 0$,

(d) $x \mapsto a\,\mathrm{f}(x) + b$, where $a < 0$.

1590. Two mechanics students speak as follows:

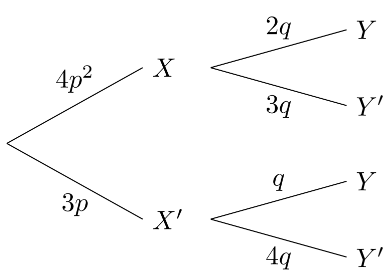
"If two particles, moving in 1D, are connected by an inextensible string, then, for $F = ma$, they can always be treated as a single object."

"That's only true if the connector is rigid."

State, with a reason, who is correct.

1591. Find the equation of the normal to the parabola $x = y^2 + 2y - 6$ at $y = -2$.

1592. Events X and Y have probabilities as represented on the following tree diagram, for constants p, q.



(a) Find p and q.

(b) Determine $\mathbb{P}(X \mid Y)$.

1593. Find constants A, B to make this an identity:

$$\frac{1}{x^4 - x^2} \equiv \frac{A}{x^2} + \frac{B}{x^2 - 1}.$$

1594. Show that $y = x^2$ and $y = 8 - (x-4)^2$ are tangent to each other.

1595. By squaring the equations and subtracting, find all possible values of R satisfying both $R \sec\theta = 13$ and $R \tan\theta = 5$.

1596. The equations $\mathrm{f}(x) = 0$ and $\mathrm{g}(x) = 0$, where f and g are quadratic functions, have the same solution set S. The equation $\mathrm{f}(x) = \mathrm{g}(x)$ is denoted E. State, with a reason, whether these claims hold:

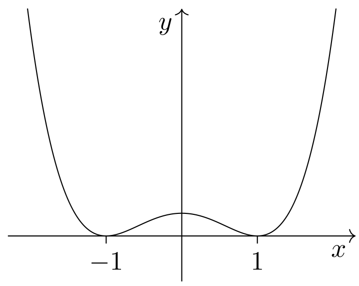
(a) "E has solution set S",

(b) "the solution set of E contains S",

(c) "the solution set of E is a subset of S".

1597. An AP starts $\sqrt{q} - 2$, $2\sqrt{q} + 1$, q. Find q.

1598. The curve shown below is $y = (x^2 - 1)^2$:



Sketch the curve $\sqrt{y} = x^2 - 1$.

1599. A student writes as follows: "The driving force on a car is really a frictional force. We call it a driving force by convention." State, with a reason, whether this is correct.

1600. Explain why the following statement is not true: "The standard deviation of a combined sample cannot be bigger than the standard deviations of the individual samples."

—— End of 16th Hundred ——